BERKSON'S THEOREM*

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ABSTRACT

We give a new proof of the theorem that Amitsur's complex for purely inseparable field extensions has vanishing homology in dimensions higher than 2. This is accomplished by computing the kernel and cokernel of the logarithmic derivative $t \to Dt/t$ mapping the multiplicative Amitsur complex to the acyclic additive one (D is a derivation of the extension field).

In [1], Amitsur introduced complexes $\mathscr{C}(F / C)$ and $\mathscr{C}^+(F / C)$ definable for any commutative algebra F with unit over a commutative ring C (with the same unit). The first has cohomology groups denoted by $H^n(F / C)$. The second has cohomology groups which usually vanish, so need no special notation.

THEOREM. *If F is a purely inseparable extension field of C then $H^n(F / C) = 0$ for $n > 2$.*

This theorem was proved in [8; Theorem 6.1] (the restriction there to finite exponent is unnecessary) by a reduction, sketched below, to the case of extensions of finite degree and exponent one. This case is then handled by a theorem of Berkson [3]. However, the same reduction reduces to the case of simple extensions of exponent one. By treating this special case explicitly, the present note achieves a slightly shorter proof than Berkson, avoids the machinery of regular, restricted Lie algebra extensions, and makes more transparent where the restriction $n > 2$ enters. S. Yuan in his Ph.D. dissertation (Northwestern University 1964) has extended the method in the present note to prove most of the results in [8] as well. Thus the appeal to [8] which we make for our reductions in fact need not appeal to any techniques (specifically, spectral sequences) less elementary than those already used here.

We begin by recalling the definitions of Amitsur's complexes. They both arise from the (co-)semisimplicial object

$$F \rightrightarrows F \otimes_C F \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} F \otimes_C F \otimes_C F \overset{\longrightarrow}{\underset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}}} \cdots$$

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We shall denote the repeated tensor product $F \otimes_C \cdots \otimes_C F$ by F^n and the maps $F^n \to F^{n+1}$ by $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n$; these maps are C-algebra homomorphisms defined by $\varepsilon_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n$ and satisfy the identities of face maps in semisimplicial complexes [7; (3.3)]. Degeneracy maps also exist, but play almost no role in the present paper. If we consider the ε's as homomorphisms of the additive groups of these F^n, we can add and subtract them to form a boundary operator $d^+ = \varepsilon_0 - \varepsilon_1 + \cdots + (-1)^n \varepsilon_n$. This forms a complex called $\mathscr{C}^+(F/C)$. Similarly the ε's induce homomorphisms of the multiplicative group $U(F^n)$ of units of F^n. The complex $\mathscr{C}(F/C)$ consists of the groups $\{U(F^n) \mid n = 1, 2, \cdots\}$ with boundary maps $d = \varepsilon_0 \cdot (1/\varepsilon_1) \cdot \varepsilon_2 \cdot \cdots \cdot (\varepsilon_n)^{\pm 1}$ from $U(F^n)$ to $U(F^{n+1})$.

We shall need a third complex $\mathscr{D}$, which is formed of the additive groups of F^2, $F^3, \cdots$ but with boundary map from F^n to F^{n+1} defined by

$$d^+ - \varepsilon_0 = -(\varepsilon_1 - \varepsilon_2 + \cdots \pm \varepsilon_n)$$

We agree to label the dimensions in $\mathscr{C}, \mathscr{D}$, and $\mathscr{C}^+$ so that the n-dimensional cochain groups are $U(F^{n+1})$, F^{n+2}, and F^{n+1} respectively. Thus in all cases the boundary of an n-cochain involves $n + 2$ ε's.

If C is a field, then $\mathscr{C}^+$ and $\mathscr{D}$ have homology groups equal to zero in all positive dimensions [7; Lemma 4.1] because a contracting homotopy may be defined by $s(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{n-2} \otimes x_{n-1}\beta(x_n)$ where $\beta: F \to C$ is any C-linear map with $\beta(1) = 1$. Furthermore, $H^0(\mathscr{C}^+)$ and $H^0(\mathscr{D})$ are isomorphic to the additive groups of C and F, respectively. In fact, $\mathscr{D}$ may be identified with $\mathscr{C}^+(F \otimes F / C \otimes F)$.

Note that in [7] the complexes $\mathscr{C}$ and $\mathscr{C}^+$ were defined to have nonzero terms in dimension -1, designed to make $\mathscr{C}^+$ acyclic in all dimensions. The present convention is somewhat more standard and more appropriate.

Since every purely inseparable extension of a field C is a direct limit of extensions of finite degree, and since $H^n(\varinjlim F_i / C) = \varinjlim H^n(F_i / C)$ (cf. [7; p. 345]), it is sufficient to prove $H^n(F/C) = 0$ when F is purely inseparable and of finite degree over C. This can in turn be reduced to the case of simple extensions $F = C(\alpha)$ with $\alpha^p \in C$ (p is the characteristic of C): Every purely inseparable F of finite degree is the top of a tower $C = F_0 \subset F_1 \subset \cdots \subset F_r = F$ with $F_{i+1} = F_i(\alpha_i)$, $\alpha_i^p \in F_i$, and by [8; Theorem 4.3] there is an exact sequence

$$H^n(F_i / C) \to H^n(F_{i+1} / C) \to H^n(F_{i+1} / F_i)$$

so that if we know $H^n(F_{i+1} / F_i) = 0$, an induction on i will prove $H^n(F/C) = 0$.

PROPOSITION. *If $F = C(\alpha)$, $\alpha^p \in C$ where F and C are fields, there is an exact sequence of complexes*

$$0 \to \varepsilon_0\mathscr{C} \to \mathscr{C}(F/C) \overset{\lambda}{\to} \mathscr{D} \overset{\gamma}{\to} \mathscr{C}^+(F/C) \to 0,$$

λ is of degree -1, γ is of degree 0, and $\varepsilon_0\mathscr{C}$, $\mathscr{D}$, and $\mathscr{C}^+(F/C)$ have homology groups equal to zero in all positive dimensions.

Corollary. $H^n(F/C) = 0$ for $n > 2$.

Proof of Corollary: The exactness of $0 \to \text{Ker } \gamma \to \mathscr{D} \to \mathscr{C}^+ \to 0$ implies $H^{n-2}(\mathscr{C}^+) \to H^{n-1}(\text{Ker } \gamma) \to H^{n-1}(\mathscr{D})$ exact. If $n > 2$, the theorem asserts that the extreme terms of the latter sequence vanish, so $H^{n-1}(\text{Ker } \gamma) = 0$. Then $0 \to \varepsilon_0\mathscr{C} \to \mathscr{C} \to \text{Ker } \gamma \to 0$ exact imples $H^n(\varepsilon_0\mathscr{C}) \to H^n(F/C) \to H^{n-1}(\text{Ker } \gamma)$ exact. Thus $H^n(F/C) = 0$.

Proof of Proposition. Let D be the derivation of F over C defined by $D(\alpha) = 1$, i.e., $D(\sum c_i\alpha^i) = \sum ic_i\alpha^{i-1}$ (this derivation has $D^p = 0$ and Ker $D = C$). Extend D to a derivation D_n of F^n over $C \otimes F^{n-1}$ by defining

$$D_n(x_1 \otimes \cdots \otimes x_n) = D(x_1) \otimes x_2 \otimes \cdots \otimes x_n.$$

Definition. If t is a unit in F^n, define $\lambda_n(t) = D_n(t)/t$; and $\lambda = \{\lambda_n \mid n = 1, 2, \cdots\}$ As in elementary calculus, $\lambda_n(tt') = \lambda_n(t) + \lambda_n(t')$; furthermore, $\lambda_{n+1}(\varepsilon_i t) = \varepsilon_i\lambda_n(t)$ for $i > 0$, and $\lambda_{n+1}(\varepsilon_0 t) = 0$ since $D(1) = 0$. Thus $\lambda: \mathscr{C}(F/C) \to \mathscr{D}$ is a homomorphism commuting with the boundary maps: $\lambda d = (d^+ - \varepsilon_0)\lambda$. That is, λ is a mapping of complexes.

To compute Ker λ we need Ker D_n, since $D_n(t)/t = 0$ if and only if $D_n(t) = 0$. Tensoring the exact sequence $0 \to \text{Ker } D \to F \overset{D}{\to} F$ with F^{n-1} we find that Ker $D_n = (\text{Ker } D) \otimes F^{n-1} = C \otimes F^{n-1} = \varepsilon_0 F^{n-1}$. Thus Ker $\lambda = \varepsilon_0\mathscr{C}$.

Now compute Im λ. If $q \in F^n$, let $L(q)$ denote the mapping $F^n \to F^n$ produced by multiplication by q. If $q = D_n(t)/t$, then

$$D_n + L(q) = L(t)^{-1}(L(t)D_n + L(D_n(t))) = L(t)^{-1}D_nL(t),$$

the last equality merely expressing the derivation property of D_n. Therefore, $(D_n + L(q))^p = L(t)^{-1}D_n{}^pL(t) = 0$ because $D^p = 0$. Conversely, if $(D_n + L(q)^p) = 0$, then Cartier proved, inter alia [1], that the ideal of F^n generated by Ker $(D_n + L(q))$ is all of F^n. In our case, F^n is a local ring (there is only one homomorphism of F^n to a field; it is the natural one $F^n \to F$), so Ker $(D_n + L(q))$ must contain a unit, u. If $t = u^{-1}$, we have $D_n(t) = -u^{-2}D_n(u) = -t^2(-qu) = qt$, so $q = D_n(t)/t$. Thus Im $\lambda_n = \{q \in F^n \mid (D_n + L(q))^p = 0\}$ However, using an old calculation ([4; Ch. 2, (36)], [5], [6]) and $D_n^p = 0$, we have

(1) $D + L(q))^p = 0$ is all that is needed to make F^n a "regular, restricted module" over the regular, restricted Lie algebra Δ of C-derivations of F. Cartier proves that every such module is FW where W is the submodule on which Δ acts as 0 [4; Chapter 2, Proposition 3]. Here $W = \text{Ker}(D + L(q))$ and F acts as $L(F \otimes C^{n-1})$, so certainly $F^n\text{Ker}(D_n + L(q)) = F^n$.

$$(D_n + L(q))^p = L(D_n^{p-1}(q) + q^p).$$

We have now shown $\operatorname{Im}\ \lambda_n = \{q \in F^n \mid D_n^{p-1}(q) + q^p = 0\}$.

DEFINITION. If $q \in F^n$, define $\gamma_n(q) = \varepsilon_0^{-1}(D_n{}^{p-1}(q) + q^p) \in F^{n-1}$, and $\gamma = \{(-1)^n\gamma_n\ n = 2,3,\cdots]$ mapping $\mathscr{D}$ to $\mathscr{C}^+$.

Note that $D_n^{p-1}(q) \in \varepsilon_0 F^{n-1}$ because $D_n(D^{p-1}(q)) = D_n^p(q) = 0$ and $\operatorname{Ker} D_n = \varepsilon_0 F^{n-1}$ as before; and that $q^p \in C \otimes \cdots \otimes C \subset C \otimes F^{n-1} = \varepsilon_0 F^{n-1}$. Thus γ_n is defined, and is single-valued because ε_0 is a monomorphism. To show γ is a mapping of complexes $\mathscr{D} \to \mathscr{C}^+$, recall that $D_n\varepsilon_i = \varepsilon_i D_n$ or 0 according as $i > 0$ or $i = 0$, so the same is true with D_n replaced by D_n^{p-1}. The simplicial identity $\varepsilon_0\varepsilon_{i-1} = \varepsilon_i\varepsilon_0$ gives $\varepsilon_{i-1}\varepsilon_0^{-1} = \varepsilon_0^{-1}\varepsilon_i$ on $\operatorname{Im}\ \varepsilon_0$, and hence $\varepsilon_{i-1}\varepsilon_0^{-1}D_n^{p-1} = \varepsilon_0^{-1}D_n^{p-1}\varepsilon_i$ for $i > 1$. Thus $d^+(\varepsilon_0^{-1}D_n^{p-1}) = (\varepsilon_0^{-1}D_n^{p-1})(\varepsilon_0 - d^+)$. Besides, since $q^p \in C^n$, $\varepsilon_i q^p = \varepsilon_0 q^p$ for all i; so $d^+(\varepsilon_0^{-1}q^p) = ((\varepsilon_0 - d^+)q)^p$. Thus both $q \to \varepsilon_0^{-1}D_n^{p-1}(q)$ and $q \to \varepsilon_0^{-1}q^p$ are mappings of $\mathscr{D}$ to $\mathscr{C}^+$ which anticommute with the boundary maps; so also does their sum. It follows that $\{(-1)^n\gamma_n\}$ commutes with the boundary map.

Since ε_0 is a monomorphism, $\operatorname{Ker}\ \gamma = \{q \in F^n \mid D_n^{p-1}(q) + q^p = 0\} = \operatorname{Im}\ \lambda_n$. To complete the exact sequence of the proposition, it remains to prove $\operatorname{Im}\ \gamma = \mathscr{C}^+$.

Given $y \in F^{n-1}$, we can produce $s \in F^n$ such that $\gamma_n(s) = \varepsilon_0^{-1}(D_n^{p-1}(s) + s^p) = y$. It suffices to take

$$s = -\alpha^{p-1} \otimes y + 1 \otimes \alpha^{p-1} y$$

where $\alpha^{p-1} \in F$ multiplies $y \in F^{n-1}$ using any of the obvious F-module structures on F^{n-1}. For then $D_n^{p-1}(s) = -(p-1)! \otimes y + 0 = 1 \otimes y = \varepsilon_0 y$ and $s^p = (-\alpha^{p-1})^p \otimes y^p + 1 \otimes (\alpha^{p-1})^p y^p = 0$ since $(\alpha^{p-1}) \in C$ and the tensor product is taken over C.

The acyclicity of $\mathscr{D}$ and $\mathscr{C}^+$was pointed out above. That $\varepsilon_0\mathscr{C}$ has vanishing homology is well-known for semisimplicial complexes. ε_0^{-1} maps the complex $\varepsilon_0\mathscr{C}$ isomorphically to a complex $\mathscr{D}'$ which is the multiplicative analog of $\mathscr{D}$, viz., the complex $\mathscr{C}$ but with boundary $\varepsilon_1(1/\varepsilon_2)\varepsilon_3\cdots$. This $\mathscr{D}'$ is acyclic because the first degeneracy $x_1 \otimes \cdots \otimes x_n \to x_1x_2 \otimes x_3 \otimes \cdots \otimes x_n$ is a contracting homotopy. This completes the proof.

REMARKS. 1. Without reference to [8] we could use the techniques in the proof of our proposition to get Berkson's full theorem, which allows any F which is purely inseparable of exponent one and finite degree. In this case $\lambda_n(t) = D_n(t)/t$ would have to be considered for all D_n induced by all derivations D of F over C; i.e., λ maps $\mathscr{C}$ into th acyclic additive complex $\{\operatorname{Hom}_F(\Delta, F_n)\}$ where Δ is the derivation algebra of F over C, thought of as a left F-module. The Cartier operator analogous to γ is actually designed to fit this more general situation.

2. By using the same computations, one can compute $H^2(F/C)$. Just as for $n>2$, $H^2(F/C)\cong H^1(\text{Ker } \gamma)$ and $H^0(\mathscr{D})\to H^0(\mathscr{C}^+)\to H^1(\text{Ker } \gamma)\to H^1(\mathscr{D})=0$ is exact. Since $H^0(\mathscr{C}^+)=C^+$ and $H^0(\mathscr{D})=F^+$ (where we denote the additive group of C by C^+), this gives the same result as in [6; Theorem 7]: $H^2(F/C)=C^+/\gamma F^+$ where $\gamma F^+=\{D^{p-1}(x)+x^p \mid x\in F\}$.

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